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Reduction of the self-dual Yang–Mills equations by Bessel and Legendre functions

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Abstract. Two integrable nonlinear equations are derived by dimensional reductions of the $SU(2)$ self-dual Yang–Mills equations. To derive the equations, we use Hirota's method and some properties of special functions, namely those of the Bessel and Legendre functions. Solutions of these equations are represented in terms of Toeplitz determinants whose elements are a superposition of the special functions.

1. Introduction

The self-dual Yang–Mills (SDYM) equations are one of the most important nonlinear systems in four dimensions not only due to their physical significance but because of their mathematical feature of complete integrability. Exact solutions of the SDYM equations have been obtained by the Riemann–Hilbert problems [1] or the Bäcklund transformations [2, 3]. Furthermore, many important soliton equations are known to be derivable from the SDYM equations by dimensional reductions and suitable choices of the gauge. This line of investigation begins with the pioneering work of Ward [4] in which he showed that the Toda lattice equation, the sine–Gordon equation, the chiral field equations, the Ernst equation, etc, could all be obtained by the reductions of the SDYM equations. Inspired by Ward's work, many other soliton equations have subsequently been shown to be derived from the SDYM equations, including the Korteweg–de Vries (KdV) and nonlinear Schrödinger (NLS) [5] equations.

In our previous paper [6], we have applied Hirota's method to the SDYM equations in an $SU(2)$ gauge theory. The bilinear forms of the SDYM equations were found to be very different from the bilinear forms that appear in the KP hierarchy. The solutions of the bilinear forms were expressed by the Toeplitz determinants whose elements satisfy linear equations called 'dispersion relations'. We also discussed some $(2+1)$-dimensional integrable equations reduced from the SDYM equations. Though Hirota's method is one of the most powerful means of obtaining exact soliton solutions explicitly, there are few works that have applied Hirota's method to the SDYM equations.

In this paper, based on Hirota's method for the SDYM equation, we discuss another type of dimensional reduction of the SDYM equations. Namely, instead of performing a dimensional reduction on the SDYM equations directly, we first derive $(1+1)$-dimensional bilinear forms. To obtain the reduced bilinear forms, we replace the four-dimensional dispersion relations of the SDYM equations with some $(1+1)$-dimensional dispersion relations by using raising and lowering operators of special functions. Then, we obtain reduced equations from the reduced bilinear forms. A similar situation is found in the reduction from the two-dimensional Toda lattice equation to the cylindrical Toda lattice equation [7]. We consider, in particular, the case of the Bessel and the Legendre functions in this paper. The reduction of using the Bessel function corresponds to doubly cylindrical reduction from the four-dimensional SDYM equation. Solutions of the reduced equations are represented in terms of Toeplitz determinants whose elements are expressed by the Bessel and the Legendre functions, respectively.

In section 2, we briefly review the bilinear forms and determinant solutions of the $SU(2)$-SDYM equations. In section 3, at first, we derive (1 + 1)-dimensional bilinear forms by using raising and lowering operators of the Bessel function. Then, we construct (1 + 1)-dimensional equations from the bilinear forms. We also give variable transformations from the four-dimensional SDYM equation to the reduced equation. The solutions of the equations are expressed by Toeplitz determinants whose elements are superpositions of the Bessel function. In section 4, we discuss the second example, i.e. the integrable reduction by the Legendre function. We give the reduced bilinear forms by using raising and lowering operators of the Legendre function. The reduced equation is derived from the bilinear forms. Section 5 is devoted to concluding remarks.

2. Self-dual Yang–Mills equation

In this section, we briefly review some properties of the SDYM equations which follow from the $SU(2)$-SDYM gauge theory.

In the $SU(2)$ gauge theory, the SDYM equations are written explicitly by using a matrix J ($\in SL(2)$) which depends on four independent variables y, $\bar{y}$, z and $\bar{z}$,

$$\partial_{\bar{y}}(J^{-1}\partial_y J) + \partial_{\bar{z}}(J^{-1}\partial_z J) = 0 \tag{1}$$

where

$$J = \frac{1}{f}\begin{pmatrix} 1 & -g \\ e & f^2 - eg \end{pmatrix}. \tag{2}$$

Our parametrization of the $SL(2)$ matrix in terms of the variables e, f, g follows the notation of Corrigan *et al* [3]. It should be noted that, for real gauge fields, we require $\bar{y} = y^*$, $\bar{z} = z^*$, $f =$ real and $g = -e^*$ (* denotes complex conjugate).

As is discussed in a previous paper [6], the bilinear forms of equation (1) are given by

$$D_y\tau_{nm} \cdot \tau_{n+1m-1} = -D_{\bar{z}}\tau_{n+1m} \cdot \tau_{nm-1} \tag{3}$$

$$D_z\tau_{nm} \cdot \tau_{n+1m-1} = D_{\bar{y}}\tau_{n+1m} \cdot \tau_{nm-1} \tag{4}$$

and

$$\tau_{n+1m}\tau_{n-1m} - \tau_{nm}^2 = \tau_{nm+1}\tau_{nm-1} \tag{5}$$

where D is Hirota's bilinear operator defined as $D_y a \cdot b = (\partial_y a)b - a(\partial_y b)$. The τ_{nm}s are $m \times m$ Toeplitz (Toda-molecule)-type determinants,

$$\tau_{nm} = \begin{vmatrix} \varphi_{n-m-1} & \cdots & \varphi_{n-2} & \varphi_{n-1} & \varphi_n \\ \varphi_{n-m} & \cdots & \varphi_{n-1} & \varphi_n & \varphi_{n+1} \\ \varphi_{n-m+1} & \cdots & \varphi_n & \varphi_{n+1} & \varphi_{n+2} \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ \varphi_n & \cdots & \varphi_{n+m-3} & \varphi_{n+m-2} & \varphi_{n+m-1} \end{vmatrix} \tag{6}$$

where n is an integer and the elements φ_j (for an integer j) are functions of the four independent variables y, $\bar{y}$, z and $\bar{z}$, which satisfy the dispersion relations,

$$\partial_y \varphi_j = -\partial_{\bar{z}} \varphi_{j+1} \tag{7}$$

$$\partial_z \varphi_j = \partial_{\bar{y}} \varphi_{j+1}. \tag{8}$$

For $m = 0$, we define $\tau_{n0} = 1$.

By using the τ_{nm}s, it is shown that equation (1) has the following series of solutions:

$$e^{(1)} = \frac{\tau_{n+1m-1}}{\tau_{nm}} \qquad f^{(1)} = \frac{\tau_{nm-1}}{\tau_{nm}} \qquad g^{(1)} = \frac{\tau_{n-1m-1}}{\tau_{nm}} \tag{9}$$

and

$$e^{(2)} = \frac{\tau_{n-1m+1}}{\tau_{nm}} \qquad f^{(2)} = \frac{\tau_{nm+1}}{\tau_{nm}} \qquad g^{(2)} = \frac{\tau_{n+1m+1}}{\tau_{nm}}. \tag{10}$$

Recasting into the matrix form, the solutions (9) and (10) each become,

$$J^{(1)}(n, m) = \frac{1}{\tau_{nm-1}} \begin{pmatrix} \tau_{nm} & -\tau_{n-1m-1} \\ \tau_{n+1m-1} & -\tau_{nm-2} \end{pmatrix} \tag{11}$$

and

$$J^{(2)}(n, m) = \frac{1}{\tau_{nm+1}} \begin{pmatrix} \tau_{nm} & -\tau_{n+1m+1} \\ \tau_{n-1m+1} & -\tau_{nm+2} \end{pmatrix} \tag{12}$$

where we have made use of Jacobi's identity (5). It is convenient to further define another series of solutions,

$$J^{(3)}(n, m) = \frac{1}{\tau_{nm}} \begin{pmatrix} \tau_{n+1m} & \tau_{nm-1} \\ \tau_{nm+1} & \tau_{n-1m} \end{pmatrix}. \tag{13}$$

It is easily found that $J^{(1)}$, $J^{(2)}$ and $J^{(3)}$ transform into one another by the relations,

$$J^{(2)}(n, m-1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} J^{(1)}(n, m+1) \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{14}$$

$$J^{(3)}(n, m) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} J^{(1)}(n, m+1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15}$$

3. Reduction case I (Bessel function)

In this section, we consider our first example of the integrable reductions of equation (1),

$$\frac{1}{\rho}\partial_\rho(\rho \tilde{P}^{-1}\partial_\rho \tilde{P}) - \frac{1}{r}\partial_r(r\tilde{P}^{-1}\partial_r \tilde{P}) = 0 \tag{16}$$

where $\tilde{P}$ is a 2×2 matrix ($\det \tilde{P} = 1$) and ρ, r are new independent variables.

Let us now derive equation (16) and discuss its determinant solutions. First, instead of the dispersion relations (7) and (8), we suppose new dispersion relations,

$$\left(\partial_\rho - \frac{j}{\rho}\right)\tilde{\varphi}_j(\rho, r) = -\left(\partial_r + \frac{j+1}{r}\right)\tilde{\varphi}_{j+1}(\rho, r) \tag{17}$$

$$\left(\partial_r - \frac{j}{r}\right)\tilde{\varphi}_j(\rho, r) = -\left(\partial_\rho + \frac{j+1}{\rho}\right)\tilde{\varphi}_{j+1}(\rho, r). \tag{18}$$

Equations (17) and (18) are reduced from equations (7) and (8) by supposing,

$$\varphi_j(y, \bar{y}, z, \bar{z}) = \left(\frac{\bar{y}}{y}\right)^{j/2} \left(\frac{\bar{z}}{-z}\right)^{j/2} \tilde{\varphi}_j(\rho, r) \tag{19}$$

where $\rho = 2\sqrt{\bar{y}y}$ and $r = 2\sqrt{\bar{z}(-z)}$. It is found that the operators which appear in equations (17) and (18) are nothing less than the raising and lowering operators of the Bessel function,

$$\left(\partial_\rho - \frac{n}{\rho}\right) J_n(\rho) = -J_{n+1}(\rho) \tag{20}$$

$$\left(\partial_\rho + \frac{n}{\rho}\right) J_n(\rho) = J_{n-1}(\rho) \tag{21}$$

where $J_n(\rho)$ is the nth Bessel function. Hence, solutions of equations (17) and (18) are expressed by using the jth Bessel function. For example,

$$\tilde{\varphi}_j(\rho, r) = \tilde{c} J_j(k\rho) J_j(kr) \tag{22}$$

is a series of solutions of equations (17) and (18), where $\tilde{c}$ and k are arbitrary constants.

Using the elements of $\tilde{\varphi}_j$, we construct $m \times m$ determinants,

$$\tilde{\tau}_{nm} \equiv \begin{vmatrix} \tilde{\varphi}_{n-m-1} & \cdots & \tilde{\varphi}_{n-2} & \tilde{\varphi}_{n-1} & \tilde{\varphi}_n \\ \tilde{\varphi}_{n-m} & \cdots & \tilde{\varphi}_{n-1} & \tilde{\varphi}_n & \tilde{\varphi}_{n+1} \\ \tilde{\varphi}_{n-m+1} & \cdots & \tilde{\varphi}_n & \tilde{\varphi}_{n+1} & \tilde{\varphi}_{n+2} \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ \tilde{\varphi}_n & \cdots & \tilde{\varphi}_{n+m-3} & \tilde{\varphi}_{n+m-2} & \tilde{\varphi}_{n+m-1} \end{vmatrix} \tag{23}$$

which have the same structure as equation (6). In a similar way to that in which equations (3) and (4) were proved in [6], we show in appendix A1 that $\tilde{\tau}_{nm}$ satisfy the following 'reduced' bilinear forms:

$$\left(D_\rho + \frac{m-n-1}{\rho}\right) \tilde{\tau}_{nm} \cdot \tilde{\tau}_{n+1m-1} = -\left(D_r + \frac{m+n}{r}\right) \tilde{\tau}_{n+1m} \cdot \tilde{\tau}_{nm-1} \tag{24}$$

$$\left(D_r + \frac{m-n-1}{r}\right) \tilde{\tau}_{nm} \cdot \tilde{\tau}_{n+1m-1} = -\left(D_\rho + \frac{m+n}{\rho}\right) \tilde{\tau}_{n+1m} \cdot \tilde{\tau}_{nm-1}. \tag{25}$$

It is easily found that $\tilde{\tau}_{nm}$ also satisfy Jacobi's identity,

$$\tilde{\tau}_{n+1m} \tilde{\tau}_{n-1m} - \tilde{\tau}_{nm}^2 = \tilde{\tau}_{nm+1} \tilde{\tau}_{nm-1}. \tag{26}$$

As the second step, we construct equation (16) from the bilinear forms (24)–(26). In appendix A2, we show that equation (16) is decomposed into the bilinear forms (24)–(26) if we choose the matrix $\tilde{P}$ as

$$\tilde{P} = \frac{1}{\tilde{\tau}_{nm}} \begin{pmatrix} (\rho r)^n \tilde{\tau}_{n+1m} & (\rho r)^{-m} \tilde{\tau}_{nm-1} \\ (\rho r)^m \tilde{\tau}_{nm+1} & (\rho r)^{-n} \tilde{\tau}_{n-1m} \end{pmatrix}. \tag{27}$$

Therefore, the determinant solutions of equation (16) are given by equation (27).

As simple examples, we explicitly give solutions for $n = 0$, $m = 0$,

$$\tilde{P} = \begin{pmatrix} 1 & 0 \\ \tilde{\varphi}_0 & 1 \end{pmatrix} \tag{28}$$

and for $n = 0$, $m = 1$,

$$\tilde{P} = \frac{1}{\tilde{\varphi}_0} \begin{pmatrix} \tilde{\varphi}_1 & (\rho r)^{-1} \\ \rho r \begin{vmatrix} \tilde{\varphi}_{-1} & \tilde{\varphi}_0 \\ \tilde{\varphi}_0 & \tilde{\varphi}_1 \end{vmatrix} & \tilde{\varphi}_{-1} \end{pmatrix} \tag{29}$$

where $\tilde{\varphi}_j$ are given by equation (22).

4. Reduction case II (Legendre function)

As the second example of integrable reduction, we consider,

$$(\xi^2-1)^{3/2}\partial_\xi\big[(\xi^2-1)^{1/2}\hat{P}^{-1}\partial_\xi\hat{P}\big]-(\eta^2-1)^{3/2}\partial_\eta\big[(\eta^2-1)^{1/2}\hat{P}^{-1}\partial_\eta\hat{P}\big]=0 \tag{30}$$

where $\hat{P}$ is a 2×2 matrix ($\det\hat{P}=1$) and ξ,η are new independent variables.

In a similar way as discussed in the previous section, we derive equation (30) and its determinant solutions. First, instead of the dispersion relations (7) and (8), we suppose new dispersion relations,

$$[(\xi^2-1)\partial_\xi+(j+1)\xi]\hat{\varphi}_j=-[(\eta^2-1)\partial_\eta-(j+1)\eta]\hat{\varphi}_{j+1} \tag{31}$$

$$[(\eta^2-1)\partial_\eta+(j+1)\eta]\hat{\varphi}_j=-[(\xi^2-1)\partial_\xi-(j+1)\xi]\hat{\varphi}_{j+1}. \tag{32}$$

Equations (31) and (32) are reduced form equations (7) and (8) by the following procedure. First, we suppose

$$\varphi_j(y,\bar{y},z,\bar{z})=\left[\frac{(\cosh\bar{y})^j(\cosh\bar{z})^j}{(\cosh y)^{(j+1)}(\cosh z)^{(j+1)}}\right]^{1/2}\hat{\varphi}_j(\xi,\eta) \tag{33}$$

with variable transformations,

$$\tanh y=-\left[\xi_0\xi+\mathrm{i}\sqrt{(\xi_0^2-1)(1-\xi^2)}\right] \tag{34}$$

$$\tanh\bar{y}=\xi_0\xi-\mathrm{i}\sqrt{(\xi_0^2-1)(1-\xi^2)} \tag{35}$$

$$\tanh z=-\left[\eta_0\eta+\mathrm{i}\sqrt{(\eta_0^2-1)(1-\eta^2)}\right] \tag{36}$$

$$\tanh\bar{z}=-\left[\eta_0\eta-\mathrm{i}\sqrt{(\eta_0^2-1)(1-\eta^2)}\right]. \tag{37}$$

Secondly, after substituting (33) into (7) and (8), we take the limits $\xi_0\to 1$, $\eta_0\to 1$. Then, we obtain equations (31) and (32).

The operators which appear in equations (31) and (32) are found to be the raising and lowering operators of the Legendre function,

$$[(\xi^2-1)\partial_\xi+(n+1)\xi]P_n(\xi)=(n+1)P_{n+1}(\xi) \tag{38}$$

$$[(\xi^2-1)\partial_\xi-n\xi]P_n(\xi)=-nP_{n-1}(\xi) \tag{39}$$

where $P_n(\xi)$ is the nth Legendre polynomial. By considering the relations (38) and (39), solutions of equations (31) and (32) are expressed by using the jth Legendre function. For example,

$$\hat{\varphi}_j=\hat{c}P_j(\xi)P_j(\eta) \tag{40}$$

is a series of solutions of equations (31) and (32), where $\hat{c}$ is an arbitrary constant.

By means of the dispersion relations (31) and (32), we can verify that the $m\times m$ determinants,

$$\hat{\tau}_{nm}=\begin{vmatrix}\hat{\varphi}_{n-m-1} & \cdots & \hat{\varphi}_{n-2} & \hat{\varphi}_{n-1} & \hat{\varphi}_n\\ \hat{\varphi}_{n-m} & \cdots & \hat{\varphi}_{n-1} & \hat{\varphi}_n & \hat{\varphi}_{n+1}\\ \hat{\varphi}_{n-m+1} & \cdots & \hat{\varphi}_n & \hat{\varphi}_{n+1} & \hat{\varphi}_{n+2}\\ \vdots & \cdots & \vdots & \vdots & \vdots\\ \hat{\varphi}_n & \cdots & \hat{\varphi}_{n+m-3} & \hat{\varphi}_{n+m-2} & \hat{\varphi}_{n+m-1}\end{vmatrix} \tag{41}$$

satisfy the bilinear forms,

$$[(\xi^2-1)D_\xi+(n-m+2)\xi]\hat{\tau}_{nm}\cdot\hat{\tau}_{n+1m-1}=[-(\eta^2-1)D_\eta+(n+m)\eta]\hat{\tau}_{n+1m}\cdot\hat{\tau}_{nm-1} \tag{42}$$

$$[(\eta^2-1)D_\eta+(n-m+2)\eta]\hat{\tau}_{nm}\cdot\hat{\tau}_{n+1m}=[-(\xi^2-1)D_\xi+(n+m)\xi]\hat{\tau}_{n+1m}\cdot\hat{\tau}_{nm-1} \tag{43}$$

and Jacobi's identity,

$$\hat{\tau}_{n+1m}\hat{\tau}_{n-1m}-\hat{\tau}_{nm}^2=\hat{\tau}_{nm+1}\hat{\tau}_{nm-1}. \tag{44}$$

In appendix B1, we give the proof of equations (42) and (43). From appendix B2, equation (30) is decomposed into the bilinear forms (42)–(44) provided that we choose

$$\hat{P}=\frac{1}{\hat{\tau}_{nm}}\begin{pmatrix} w^{-(2n+1)/4}\hat{\tau}_{n+1m} & w^{(2m-1)/4}\hat{\tau}_{nm-1} \\ w^{-(2m-1)/4}\hat{\tau}_{nm+1} & w^{(2n+1)/4}\hat{\tau}_{n-1m}\end{pmatrix} \tag{45}$$

where $w=(\xi^2-1)(\eta^2-1)$.

As simple examples, we explicitly give solutions for $n=0$, $m=0$,

$$\hat{P}=\begin{pmatrix} w^{-1/4} & 0 \\ w^{1/4}\hat{\varphi}_0 & w^{1/4}\end{pmatrix} \tag{46}$$

and for $n=0$, $m=1$,

$$\hat{P}=\frac{1}{\hat{\varphi}_0}\begin{pmatrix} w^{-1/4}\hat{\varphi}_1 & w^{1/4} \\ w^{-1/4}\begin{vmatrix}\hat{\varphi}_{-1} & \hat{\varphi}_0 \\ \hat{\varphi}_0 & \hat{\varphi}_1\end{vmatrix} & w^{1/4}\hat{\varphi}_{-1}\end{pmatrix} \tag{47}$$

where $\hat{\varphi}_j$ are given by equation (40).

5. Concluding remarks

In this paper, we have discussed the integrable reductions of the $SU(2)$ self-dual Yang–Mills equations. The key features of our reduction was the use of Hirota's method and the raising and lowering operators of the Bessel and Legendre functions. The reduction resulted in new integrable equations, the solutions of which were represented by Toeplitz determinants whose elements were superpositions of the Bessel and Legendre functions, respectively.

Though we specialize to the Bessel and Legendre functions in the present study, the method developed here may be applicable to other special functions to obtain integrable equations. We will discuss this subject elsewhere.

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Appendix A1

In this appendix, we give a proof for equation (24). Equation (25) can also be proved in the same way. First, we introduce a dummy variable λ and define

$$\tilde{\varphi}'_j\equiv\tilde{\varphi}_j(\rho,r)\,\mathrm{e}^{j\lambda} \tag{A1}$$

and

$$\tilde{\tau}'_{nm} \equiv \begin{vmatrix} \tilde{\varphi}'_{n-m-1} & \tilde{\varphi}'_{n-m} & \cdots & \tilde{\varphi}'_{n} \\ \tilde{\varphi}'_{n-m} & \tilde{\varphi}'_{n-m+1} & \cdots & \tilde{\varphi}'_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\varphi}'_{n} & \tilde{\varphi}'_{n+1} & \cdots & \tilde{\varphi}'_{n+m-1} \end{vmatrix} = \tilde{\tau}_{nm}\,\mathrm{e}^{mn\lambda}. \tag{A2}$$

By using equation (A1), we rewrite equation (17) as

$$\left(\partial_\rho - \frac{1}{\rho}\partial_\lambda\right)\tilde{\varphi}'_j = -\mathrm{e}^{-\lambda}\left(\partial_r + \frac{1}{r}\partial_\lambda\right)\tilde{\varphi}'_{j+1} \tag{A3}$$

and we easily find that

$$\mathrm{e}^{-mn\lambda}\partial_\lambda\tilde{\tau}'_{nm} = mn\tilde{\tau}_{nm}. \tag{A4}$$

We already know that equation (3) holds provided that the elements of determinants (6) satisfy the dispersion relation (7). Therefore, if we replace the dispersion relation (3) with (A3), by using equation (A4), we find that the left-hand side of equation (24) is transformed into

$$\begin{aligned}\left[D_\rho + \frac{m-n-1}{\rho}\right]\tilde{\tau}_{nm}\cdot\tilde{\tau}_{n+1m-1} &= \left[D_\rho - \frac{mn-(m-1)(n+1)}{\rho}\right]\tilde{\tau}_{nm}\cdot\tilde{\tau}_{n+1m-1} \\ &= \mathrm{e}^{-mn\lambda}\mathrm{e}^{-(m-1)(n+1)\lambda}\left[D_\rho - \frac{1}{\rho}D_\lambda\right]\tilde{\tau}'_{nm}\cdot\tilde{\tau}'_{n+1m-1} \\ &= -\mathrm{e}^{-mn\lambda}\mathrm{e}^{-(m-1)(n+1)\lambda}\mathrm{e}^{-\lambda}\left[D_r + \frac{1}{r}D_\lambda\right]\tilde{\tau}'_{n+1m}\cdot\tilde{\tau}'_{nm-1} \\ &= -\left[D_r + \frac{m(n+1)-(m-1)n}{r}\right]\tilde{\tau}_{n+1m}\cdot\tilde{\tau}_{nm-1}. \end{aligned} \tag{A5}$$

Hence, equation (24) is proved.

Appendix A2

Here we show how to decompose equation (16) into the bilinear forms (24)–(26). First, we define,

$$\begin{array}{lll} \alpha = (\rho r)^n\tilde{\tau}_{n+1m} & \beta = (\rho r)^{-m}\tilde{\tau}_{nm-1} & \gamma = (\rho r)^m\tilde{\tau}_{nm+1} \\ \delta = (\rho r)^{-n}\tilde{\tau}_{n-1m} & \epsilon = (\rho r)^{n+m+1}\tilde{\tau}_{n+1m+1} & \zeta = (\rho r)^{-n+m+1}\tilde{\tau}_{n-1m+1} \\ \theta = (\rho r)^{n-m+1}\tilde{\tau}_{n+1m-1} & \kappa = (\rho r)^{-n-m+1}\tilde{\tau}_{n-1m-1} & \mu = \tilde{\tau}_{nm}. \end{array} \tag{A6}$$

Then, from equations (24) and (25), it is found that

$$D_\rho\mu\cdot\theta = -(\rho r)D_r\alpha\cdot\beta \tag{A7}$$

$$D_r\mu\cdot\theta = -(\rho r)D_\rho\alpha\cdot\beta \tag{A8}$$

$$(\rho r)D_\rho\gamma\cdot\alpha = -D_r\epsilon\cdot\mu \tag{A9}$$

$$(\rho r)D_r\gamma\cdot\alpha = -D_\rho\epsilon\cdot\mu \tag{A10}$$

$$(\rho r)D_\rho\delta\cdot\beta = -D_r\mu\cdot\kappa \tag{A11}$$

$$(\rho r)D_r\delta\cdot\beta = -D_\rho\mu\cdot\kappa \tag{A12}$$

$$D_\rho\zeta\cdot\mu = -(\rho r)D_r\gamma\cdot\delta \tag{A13}$$

$$D_r\zeta\cdot\mu = -(\rho r)D_\rho\gamma\cdot\delta. \tag{A14}$$

Secondly, let us suppose,

$$\tilde{P} = \frac{1}{\mu}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{A15}$$

Then, after some lengthy calculations, we find that each component of equation (16) is decomposed into the following bilinear forms:

$$\begin{aligned}
\rho r\left[\frac{1}{\rho}\partial_\rho(\rho\tilde{P}^{-1}\partial_\rho\tilde{P}) - \frac{1}{r}\partial_r(r\tilde{P}^{-1}\partial_r\tilde{P})\right]_{11} &= \frac{1}{4}\left(\frac{\beta}{\alpha}+\frac{\delta}{\gamma}\right)\left\{\partial_\rho\left[\frac{1}{\mu^2}(\rho r D_\rho\alpha\cdot\gamma - D_r\epsilon\cdot\mu)\right]\right. \\
&\quad \left. -\partial_r\left[\frac{1}{\mu^2}(\rho r D_r\alpha\cdot\gamma - D_\rho\epsilon\cdot\mu)\right]\right\} \\
&\quad + \frac{1}{4}\left(\frac{\alpha}{\beta}+\frac{\gamma}{\delta}\right)\left\{\partial_\rho\left[\frac{1}{\mu^2}(\rho r D_\rho\beta\cdot\delta - D_r\mu\cdot\kappa)\right]\right. \\
&\quad \left. -\partial_r\left[\frac{1}{\mu^2}(\rho r D_r\beta\cdot\delta - D_\rho\mu\cdot\kappa)\right]\right\} \\
&\quad + \frac{1}{4}\left(\frac{\delta}{\beta}-\frac{\gamma}{\alpha}\right)\left\{\partial_\rho\left[\frac{1}{\mu^2}(\rho r D_\rho\alpha\cdot\beta + D_r\mu\cdot\theta)\right]\right. \\
&\quad \left. -\partial_r\left[\frac{1}{\mu^2}(\rho r D_r\alpha\cdot\beta + D_\rho\mu\cdot\theta)\right]\right\} \\
&\quad + \frac{1}{4}\left(\frac{\alpha}{\gamma}-\frac{\beta}{\delta}\right)\left\{\partial_\rho\left[\frac{1}{\mu^2}(\rho r D_\rho\gamma\cdot\delta + D_r\zeta\cdot\mu)\right]\right. \\
&\quad \left. -\partial_r\left[\frac{1}{\mu^2}(\rho r D_r\gamma\cdot\delta + D_\rho\zeta\cdot\mu)\right]\right\} \\
&\quad + \left(\partial_\rho\frac{\rho r}{2\mu^2}\partial_\rho - \partial_r\frac{\rho r}{2\mu^2}\partial_r\right)[\alpha\delta - \beta\gamma - \mu^2]
\end{aligned} \tag{A16}$$

$$\begin{aligned}
\rho r\left[\frac{1}{\rho}\partial_\rho(\rho\tilde{P}^{-1}\partial_\rho\tilde{P}) - \frac{1}{r}\partial_r(r\tilde{P}^{-1}\partial_r\tilde{P})\right]_{12} &= -\partial_\rho\left[\frac{1}{\mu^2}(\rho r D_\rho\delta\cdot\beta + D_r\mu\cdot\kappa)\right] \\
&\quad +\partial_r\left[\frac{1}{\mu^2}(\rho r D_r\delta\cdot\beta + D_\rho\mu\cdot\kappa)\right]
\end{aligned} \tag{A17}$$

$$\begin{aligned}
\rho r\left[\frac{1}{\rho}\partial_\rho(\rho\tilde{P}^{-1}\partial_\rho\tilde{P}) - \frac{1}{r}\partial_r(r\tilde{P}^{-1}\partial_r\tilde{P})\right]_{21} &= \partial_\rho\left[\frac{1}{\mu^2}(\rho r D_\rho\gamma\cdot\alpha + D_r\epsilon\cdot\mu)\right] \\
&\quad -\partial_r\left[\frac{1}{\mu^2}(\rho r D_r\gamma\cdot\alpha + D_\rho\epsilon\cdot\mu)\right]
\end{aligned} \tag{A18}$$

and

$$\begin{aligned}
\rho r\left[\frac{1}{\rho}\partial_\rho(\rho\tilde{P}^{-1}\partial_\rho\tilde{P}) - \frac{1}{r}\partial_r(r\tilde{P}^{-1}\partial_r\tilde{P})\right]_{22} &= -[\partial_\rho(\rho r\tilde{P}^{-1}\partial_\rho\tilde{P}) - \partial_r(\rho r\tilde{P}^{-1}\partial_r\tilde{P})]_{11} \\
&\quad + \left(\partial_\rho\frac{\rho r}{\mu^2}\partial_\rho - \partial_r\frac{\rho r}{\mu^2}\partial_r\right)[\alpha\delta - \beta\gamma - \mu^2].
\end{aligned} \tag{A19}$$

Therefore, it is proved that equation (16) is decomposed into the bilinear forms (A7)–(A14), or equivalently, equations (24) and (25).

Appendix B1

Here, we give a proof for equation (42). Equation (43) can be also proved in the same way.

First, with the dummy variable λ, we define

$$\hat{\varphi}'_j \equiv \hat{\varphi}_j(\xi, \eta)\,\mathrm{e}^{(j+1)\lambda} \tag{B1}$$

and

$$\hat{\tau}'_{nm} \equiv \begin{vmatrix} \hat{\varphi}'_{n-m-1} & \hat{\varphi}'_{n-m} & \cdots & \hat{\varphi}'_{n} \\ \hat{\varphi}'_{n-m} & \hat{\varphi}'_{n-m+1} & \cdots & \hat{\varphi}'_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\varphi}'_{n} & \hat{\varphi}'_{n+1} & \cdots & \hat{\varphi}'_{n+m-1} \end{vmatrix} = \hat{\tau}_{nm}\mathrm{e}^{m(n+1)\lambda}. \tag{B2}$$

By using equation (B1), we rewrite equation (31) as

$$[(\xi^2-1)\partial_\xi + \xi\partial_\lambda]\hat{\varphi}'_j = -\mathrm{e}^{-\lambda}[(\eta^2-1)\partial_\eta + \eta - \eta\partial_\lambda]\hat{\varphi}'_{j+1} \tag{B3}$$

and we easily find that

$$\mathrm{e}^{-m(n+1)\lambda}\partial_\lambda\hat{\tau}'_{nm} = m(n+1)\hat{\tau}_{nm}. \tag{B4}$$

In a similar way to that discussed in appendix A1, by equations (B3) and (B4), the left-hand side of equation (42) is transformed into

$$\begin{aligned}
&[(\xi^2-1)D_\xi + (n-m+2)\xi]\hat{\tau}_{nm}\cdot\hat{\tau}_{n+1m-1} \\
&\quad= [(\xi^2-1)D_\xi + m(n-1)\xi - (m-1)(n+2)\xi]\hat{\tau}_{nm}\cdot\hat{\tau}_{n+1m-1} \\
&\quad= \mathrm{e}^{-m(n+1)\lambda}\mathrm{e}^{-(m-1)(n+2)\lambda}[(\xi^2-1)D_\xi + \xi D_\lambda]\hat{\tau}'_{nm}\cdot\hat{\tau}'_{n+1m-1} \\
&\quad= -\mathrm{e}^{-m(n+1)\lambda}\mathrm{e}^{-(m-1)(n+2)\lambda}\mathrm{e}^{-\lambda}\{\hat{\tau}'_{nm-1}[(\eta^2-1)\partial_\eta + m\eta - \eta\partial_\lambda]\hat{\tau}'_{n+1m} \\
&\qquad -\hat{\tau}'_{n+1m}[(\eta^2-1)\partial_\eta + (m-1)\eta - \eta\partial_\lambda]\hat{\tau}'_{nm-1}\} \\
&\quad= -[(\eta^2-1)D_\eta - \eta(m+n)]\hat{\tau}_{n+1m}\cdot\hat{\tau}_{nm-1}.
\end{aligned} \tag{B5}$$

Then, equation (42) is proved.

Appendix B2

We show how to decompose equation (30) into the bilinear forms (42)–(44). We define new variables,

$$\begin{aligned}
&\hat{\alpha} = w^{-\frac{1}{2}n-\frac{1}{4}}\hat{\tau}_{n+1m} && \hat{\beta} = w^{\frac{1}{2}m-\frac{1}{4}}\hat{\tau}_{nm-1} && \hat{\gamma} = w^{-\frac{1}{2}m+\frac{1}{4}}\hat{\tau}_{nm+1} \\
&\hat{\delta} = w^{\frac{1}{2}n+\frac{1}{4}}\hat{\tau}_{n-1m} && \hat{\epsilon} = w^{-\frac{1}{2}(m+n+1)}\hat{\tau}_{n+1m+1} && \hat{\zeta} = w^{-\frac{1}{2}(m-n)}\hat{\tau}_{n-1m+1} \\
&\hat{\theta} = w^{\frac{1}{2}(m-n-2)}\hat{\tau}_{n+1m-1} && \hat{\kappa} = w^{\frac{1}{2}(m+n-1)}\hat{\tau}_{n-1m-1} && \hat{\mu} = \hat{\tau}_{nm}
\end{aligned} \tag{B6}$$

where $w = (\xi^2-1)(\eta^2-1)$. Then, from equations (42) and (43), it is found that the new variables satisfy the following bilinear forms:

$$(\xi^2-1)D_\xi\hat{\mu}\cdot\hat{\theta} = -w^{-1/2}(\eta^2-1)D_\eta\hat{\alpha}\cdot\hat{\beta} \tag{B7}$$

$$(\eta^2-1)D_\eta\hat{\mu}\cdot\hat{\theta} = -w^{-1/2}(\xi^2-1)D_\xi\hat{\alpha}\cdot\hat{\beta} \tag{B8}$$

$$w^{-1/2}(\xi^2-1)D_\xi\hat{\gamma}\cdot\hat{\alpha} = -(\eta^2-1)D_\eta\hat{\epsilon}\cdot\hat{\mu} \tag{B9}$$

$$w^{-1/2}(\eta^2-1)D_\eta\hat{\gamma}\cdot\hat{\alpha} = -(\xi^2-1)D_\xi\hat{\epsilon}\cdot\hat{\mu} \tag{B10}$$

$$w^{-1/2}(\xi^2-1)D_\xi\hat{\delta}\cdot\hat{\beta} = -(\eta^2-1)D_\eta\hat{\mu}\cdot\hat{\kappa} \tag{B11}$$

$$w^{-1/2}(\eta^2-1)D_\eta\hat{\delta}\cdot\hat{\beta} = -(\xi^2-1)D_\xi\hat{\mu}\cdot\hat{\kappa} \tag{B12}$$

$$(\xi^2-1)D_\xi\hat{\zeta}\cdot\hat{\mu} = -w^{-1/2}(\eta^2-1)D_\eta\hat{\gamma}\cdot\hat{\delta} \tag{B13}$$

$$(\eta^2-1)D_\eta\hat{\zeta}\cdot\hat{\mu} = -w^{-1/2}(\xi^2-1)D_\xi\hat{\gamma}\cdot\hat{\delta}. \tag{B14}$$

By using the newly introduced variables, if we suppose

$$\hat{P} = \frac{1}{\hat{\mu}} \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \hat{\gamma} & \hat{\delta} \end{pmatrix} \tag{B15}$$

then, we find that each component of equation (30) is decomposed into the following bilinear forms:

$$\begin{aligned}
&w^{1/2}[(\xi^2-1)\partial_\xi(w^{-1/2}(\xi^2-1)\hat{P}^{-1}\partial_\xi\hat{P}) - (\eta^2-1)\partial_\eta(w^{-1/2}(\eta^2-1)\hat{P}^{-1}\partial_\eta\hat{P})]_{11}\\
&\quad = \frac{w^{1/2}}{4}\left(\frac{\hat{\beta}}{\hat{\alpha}} + \frac{\hat{\delta}}{\hat{\gamma}}\right)\left\{(\xi^2-1)\partial_\xi\left[\frac{1}{\hat{\mu}^2}[w^{-1/2}(\xi^2-1)D_\xi\hat{\alpha}\cdot\hat{\gamma} - (\eta^2-1)D_\eta\epsilon\cdot\hat{\mu}]\right]\right.\\
&\qquad \left.-(\eta^2-1)\partial_\eta\left[\frac{1}{\hat{\mu}^2}[w^{-1/2}(\eta^2-1)D_\eta\hat{\alpha}\cdot\hat{\gamma} - (\xi^2-1)D_\xi\epsilon\cdot\hat{\mu}]\right]\right\}\\
&\qquad + \frac{w^{1/2}}{4}\left(\frac{\hat{\alpha}}{\hat{\beta}} + \frac{\hat{\gamma}}{\hat{\delta}}\right)\\
&\qquad \times\left\{(\xi^2-1)\partial_\xi\left[\frac{1}{\hat{\mu}^2}[w^{-1/2}(\xi^2-1)D_\xi\hat{\beta}\cdot\hat{\delta} - (\eta^2-1)D_\eta\hat{\mu}\cdot\kappa]\right]\right.\\
&\qquad \left.-(\eta^2-1)\partial_\eta\left[\frac{1}{\hat{\mu}^2}[w^{-1/2}(\eta^2-1)D_\eta\hat{\beta}\cdot\hat{\delta} - (\xi^2-1)D_\xi\hat{\mu}\cdot\kappa]\right]\right\}\\
&\qquad + \frac{w^{1/2}}{4}\left(\frac{\hat{\delta}}{\hat{\beta}} - \frac{\hat{\gamma}}{\hat{\alpha}}\right)\\
&\qquad \times\left\{(\xi^2-1)\partial_\xi\left[\frac{1}{\hat{\mu}^2}[w^{-1/2}(\xi^2-1)D_\xi\hat{\alpha}\cdot\hat{\beta} + (\eta^2-1)D_\eta\hat{\mu}\cdot\theta]\right]\right.\\
&\qquad \left.-(\eta^2-1)\partial_\eta\left[\frac{1}{\hat{\mu}^2}[w^{-1/2}(\eta^2-1)D_\eta\hat{\alpha}\cdot\hat{\beta} + (\xi^2-1)D_\xi\hat{\mu}\cdot\theta]\right]\right\}\\
&\qquad + \frac{w^{1/2}}{4}\left(\frac{\hat{\alpha}}{\hat{\gamma}} - \frac{\hat{\beta}}{\hat{\delta}}\right)\\
&\qquad \times\left\{(\xi^2-1)\partial_\xi\left[\frac{1}{\hat{\mu}^2}[w^{-1/2}(\xi^2-1)D_\xi\hat{\gamma}\cdot\hat{\delta} + (\eta^2-1)D_\eta\zeta\cdot\hat{\mu}]\right]\right.\\
&\qquad \left.-(\eta^2-1)\partial_\eta\left[\frac{1}{\hat{\mu}^2}[w^{-1/2}(\eta^2-1)D_\eta\hat{\gamma}\cdot\hat{\delta} + (\xi^2-1)D_\xi\hat{\mu}\zeta\hat{\mu}]\right]\right\}\\
&\qquad + w^{1/2}(\xi^2-1)\partial_\xi\left[\frac{w^{-1/2}(\xi^2-1)}{2\hat{\mu}^2}\partial_\xi(\hat{\alpha}\hat{\delta} - \hat{\beta}\hat{\gamma} - \hat{\mu}^2)\right]\\
&\qquad + w^{1/2}(\eta^2-1)\partial_\eta\left[\frac{w^{-1/2}(\eta^2-1)}{2\hat{\mu}^2}\partial_\eta(\hat{\alpha}\hat{\delta} - \hat{\beta}\hat{\gamma} - \hat{\mu}^2)\right]
\end{aligned} \tag{B16}$$

$$\begin{aligned}
&w^{1/2}[(\xi^2-1)\partial_\xi(w^{-1/2}(\xi^2-1)\hat{P}^{-1}\partial_\xi\hat{P}) - (\eta^2-1)\partial_\eta(w^{-1/2}(\eta^2-1)\hat{P}^{-1}\partial_\eta\hat{P})]_{12}\\
&\quad = w^{1/2}\left\{(\xi^2-1)\partial_\xi\left[\frac{1}{\hat{\mu}^2}[w^{-1/2}(\xi^2-1)D_\xi\hat{\beta}\cdot\hat{\delta} - (\eta^2-1)D_\eta\hat{\mu}\cdot\kappa]\right]\right.\\
&\qquad \left.-(\eta^2-1)\partial_\eta\left[\frac{1}{\hat{\mu}^2}[w^{-1/2}(\eta^2-1)D_\eta\hat{\beta}\cdot\hat{\delta} - (\xi^2-1)D_\xi\hat{\mu}\cdot\kappa]\right]\right\}
\end{aligned} \tag{B17}$$

$$w^{1/2}[(\xi^2-1)\partial_\xi(w^{-1/2}(\xi^2-1)\hat{P}^{-1}\partial_\xi\hat{P})-(\eta^2-1)\partial_\eta(w^{-1/2}(\eta^2-1)\hat{P}^{-1}\partial_\eta\hat{P})]_{21}$$
$$= w^{1/2}\left\{(\xi^2-1)\partial_\xi\left[\frac{1}{\hat{\mu}^2}[w^{-1/2}(\xi^2-1)D_\xi\hat{\gamma}\cdot\hat{\alpha}-(\eta^2-1)D_\eta\epsilon\cdot\hat{\mu}]\right]\right.$$
$$\left.-(\eta^2-1)\partial_\eta\left[\frac{1}{\hat{\mu}^2}[w^{-1/2}(\eta^2-1)D_\eta\hat{\gamma}\cdot\hat{\alpha}-(\xi^2-1)D_\xi\epsilon\cdot\hat{\mu}]\right]\right\} \quad \text{(B18)}$$

and

$$w^{1/2}[(\xi^2-1)\partial_\xi(w^{-1/2}(\xi^2-1)\hat{P}^{-1}\partial_\xi\hat{P})-(\eta^2-1)\partial_\eta(w^{-1/2}(\eta^2-1)\hat{P}^{-1}\partial_\eta\hat{P})]_{22}$$
$$= -w^{1/2}[(\xi^2-1)\partial_\xi(w^{-1/2}(\xi^2-1)\hat{P}^{-1}\partial_\xi\hat{P})$$
$$-(\eta^2-1)\partial_\eta(w^{-1/2}(\eta^2-1)\hat{P}^{-1}\partial_\eta\hat{P})]_{11}$$
$$+w^{1/2}(\xi^2-1)\partial_\xi\left[\frac{w^{-1/2}(\xi^2-1)}{\hat{\mu}^2}\partial_\xi(\hat{\alpha}\hat{\delta}-\hat{\beta}\hat{\gamma}-\hat{\mu}^2)\right]$$
$$+w^{1/2}(\eta^2-1)\partial_\eta\left[\frac{w^{-1/2}(\eta^2-1)}{\hat{\mu}^2}\partial_\eta(\hat{\alpha}\hat{\delta}-\hat{\beta}\hat{\gamma}-\hat{\mu}^2)\right]. \quad \text{(B19)}$$

Therefore, equation (30) follows from the bilinear forms (B7)–(B14), or equivalently, equations (42)–(44).

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